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ABSTRACT

Teachers' perceptions about the nature of their discipline are important factors in their curricular decisions, behavior in the classroom, and interpretations and reactions to classroom events. It is not an easy task to access one's beliefs nor stimulate genuine dialogue about them. Errors can provide both the motivations and the means for engaging in valuable mathematical explorations and reflections about the nature of mathematics as a guide for an analysis of the subject. The first part of the paper concerns the rationale behind the use of errors. Then a report and an in-depth analysis of a specific experience conducted in a graduate mathematics course is discussed along with a brief description of additional units organized around the study of error. Sections included in the paper are: (1) "Introduction"; (2) "Rationale of the Strategy"; (3) "Using Errors as Springboards for Reflections on the Nature of Mathematics"; (4) "Analyzing Incorrect Definitions of 'Circle': An Illustration"; (5) "Other Applications of the Strategy"; and (6) "Conclusions." References are listed. (RT)

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USING ERRORS TO STIMULATE A DEBATE ON THE NATURE OF MATHEMATICS: EXPERIENCES IN TEACHER EDUCATION COURSES

by

Raffaella Borasi

Paper presented at the Annual Meeting of the

American Educational Research Association

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USING ERRORS TO STIMULATE A DEBATE ON THE NATURE OF MATHEMATICS: EXPERIENCES IN TEACHER EDUCATION COURSES

Raffaella Borasi - University of Rochester

I - Introduction

Recent research on teaching has made us aware of the key role of teachers' beliefs about the nature of their discipline, both for their curricular decisions and their behavior in the classroom. At the same time, we have learned to appreciate that one's beliefs are not easy to access, nor is it simple to stimulate genuine dialogue about them. In this paper, I suggest that mathematical errors could provide a valuable means to engage teachers in reflections on the nature of mathematics which will elicit their own beliefs on the subject as well as open them to the possibility of alternatives.

The first part of the paper will be devoted to the presentation of the rationale behind such use of errors -- by discussing the role of teachers' beliefs on the nature of mathematics, and the potential of mathematical errors to encourage reflections on various aspects of the discipline, respectively. To better illustrate the potential of the strategy proposed for teacher education, I will then report and analyze in-depth a specific experience conducted in a graduate mathematics education course, where the analysis of a list of incorrect definitions

of 'circle' led the participants to challenge some of their original expectations about the notion of 'mathematical definition'. A brief description of other units organized around the study of an error will then follow, to provide an idea of the variety of ways in which the strategy can be implemented.

Although the experiences reported in this paper have been developed in the context of mathematics (since this is the author's area of expertise), I believe that the strategy illustrated could be of interest to a larger audience than mathematics educators, since appropriate errors could be used to encourage any teacher to reflect on the nature, strength and limitations of his/her own discipline.

II - Rationale of the Strategy

Dealing with Teachers' Beliefs on the Nature of Mathematics

It seems reasonable to assume that the "lens" through which an individual sees the world is an important factor in how that individual interprets and reacts to events around him/her. Kuhn has argued forcefully how this can affect the work of a scientist [Kuhn, 1970]. Similarly, we can assume that the subjective theories about the nature of mathematics held by a teacher will determine much of what s/he defines as the problems of daily teaching.

For example, suppose a teacher believes that mathematics consists essentially of a set of pre-determined rules and procedures to be mastered. Then it seems reasonable that she will focus her energies in preparing clear explanations and well-sequenced sets of application exercises, and will evaluate her students' performance mainly in terms of their ability to produce "correct answers" to the exercises and problems assigned to them. On the contrary, a teacher who perceives that the essence of mathematics lies in the employment of logical deduction to derive the consequences of a given set of assumptions is more likely to emphasize proofs and justifications versus correct results, both in her teaching and in the evaluation of her students' work. While these two example may seem rather extreme and obvious, recent research studies on teachers' beliefs have revealed that a teacher's conception of his/her discipline can also influence his/her

decision making also in more pervasive and subtle ways [Cooney and Brown, 1982; Brown, Brown, Cooney and Smith, 1982; Cooney, 1985].

Though the results from studies assessing the belief system held by individual mathematics teachers are almost impossible to generalize, they have at least suggested that many teachers share with students and laymen a dualistic conception of mathematics [Meyerson, 1977; Copes, 1982; Cooney and Brown, 1982; Cooney, 1985]. Mathematics is thus perceived as a rigid and impersonal discipline, where results are always uniquely determined and either absolutely right or wrong, and where there is no space for personal judgement and values. Philosophers of science (such as Kuhn [1970] and Lakatos [1976]), historians of mathematics (such as Kline [1980]) and mathematics educators (such as Brown [1982]), on the contrary, have challenged this conception of mathematics. They have pointed out that the growth of mathematical knowledge has in fact been marked with errors, uncertainty, and struggles between opposing viewpoints -- the debate generated around the parallel postulate and the consequent development of alternative non-Euclidean geometries provides a remarkable example. Unproved conjectures and open problems are alive and well in all branches of mathematics, as are limitations and contradictions which mathematicians have had to accept and live with -- we can find evidence to this regard in the problems encountered dealing with infinity even at an elementary level, besides of course the problems made famous by Godel and other logicians within the foundations of mathematics. Professional mathematicians often rely on intuition in their proofs, and many have argued that whether a justification can be considered sufficient depends

essentially upon the agreement of a specific scientific community, rather than on the rigid application of a set of logic rules.

In light of these considerations, therefore, it seems that what characterizes mathematics as a discipline is more controversial than what it might have appeared at first and that mathematics teachers may greatly gain from engaging in a discussion on the nature of mathematics.

However, there are several problems connected with bringing teachers to explicitly examine and reconsider such subjective theories of mathematics. First of all, given the highly technical emphasis of their mathematical training, teachers have seldom (if ever) addressed issues regarding the nature of mathematics within previous mathematics courses, and they may not even recognize the possibility of controversy on such issues. Secondly, it may be difficult to explicitly discuss issues of a high level of abstraction and sophistication such as the role of rigor or the nature of truth in mathematics. Thirdly, as researchers have argued (cf. Brown and Cooney, 1982), a person may not always be aware of the beliefs s/he holds, and thus individual conceptions of mathematics need to be "captured" in more indirect ways.

To overcome these problems, mathematics educators should then seek to create in their courses "rich" situations that indirectly and spontaneously elicit the expression of teachers' beliefs, and consequently stimulate dialogue amongst the participants, rather than initiate open discussions on the nature of mathematics. I suggest that errors have the potential to create such situations -- though they are obviously not the unique way to accomplish this goal! -- and I will elaborate on this claim in the next section.

Using Errors as Springboards for Reflections on the Nature of mathematics

Elsewhere I have argued that errors can provide both the motivation and the means to engage in valuable mathematical explorations and reflections on the nature of mathematics (Borasi, 1986a, 1987b, 1988a). Because errors signify that our original expectations have not been met, they naturally invite some action on our part. Since they represent a contrast with what we originally expected, errors can also provide a concrete starting point and a guide for our analysis.

To fully appreciate this educational potential of mathematical errors, we must first realize that the errors which may occur in mathematics are of considerable variety. Many people would tend to identify mathematical errors with incorrect results due to the misuse of rules or procedures. On the contrary, errors can also occur in many other contexts and present varying degrees of "wrongness", as illustrated by the following examples:

(1) The conjecture that 1/a-1/b=1/ab, based on the following data:
1/2 - 1/3 = 1/6
1/3 - 1/4 = 1/12
1/6 - 1/7 = 1/42.

(2) The acceptance of the result: $\sqrt{2+\sqrt{2+\ldots+\sqrt{2+\sqrt{2}}}} = 2$ on the basis of the following procedure (or "proof"):

$\sqrt{2+\sqrt{2+\ldots+\sqrt{2+\sqrt{2}}}} = x$ ---> $2 + \sqrt{2+\ldots+\sqrt{2+\sqrt{2}}} = x^2$ --->
$2 + x = x^2$ ---> $x = 2$ or $x = -1$; $x>0$ because it is a radical --->
$x = \sqrt{2+\sqrt{2+\ldots+\sqrt{2+\sqrt{2}}}} = 2$.

(3) The following definitions of circle:
"A collection of points all equidistant from the center (radius)"
"A closed, continuous, rounded line"
"A straight line that changes direction constantly"

(4) The expectation that one will obtain "head" 50 times when tossing a coin 100 times.

These examples suggest that mathematical errors can be used to initiate reflections on the nature of mathematics, and gain a better understanding of the discipline, at least along two complementary directions:

a) General mathematical notions -- such as "proof", "definition" or "model" -- could be investigated through the analysis of the kind of errors which could occur with regard to each of them. By contrasting erroneous and correct instances of definitions for a certain concept, for example, we may be able to better identify and formulate the criteria required in mathematics to create "good" definitions. At the same time, we may also discover limitations inherent to the notion of mathematical definition per-se which may in turn affect our expectations and use of definitions in the future.

b) The consideration of "borderline errors" such as partial, approximate, or probabilistic results can help us realize that in mathematics, as well as in most other areas of knowledge, results are not always either right or wrong and their evaluation may depend considerably on the context in which we are operating. For example, attempting to "fix up" a somewhat unrigorous proof can lead us to question the extent to which we can and should explicitly state all the steps in the proof, what makes a proof understandable and convincing, and whether the two desirable elements of rigor and clarity may sometimes be conflicting. This analysis of errors may both initiate and guide a reconsideration of the nature of truth in

mathematics, as well as of our expectations in terms of rigor and precision.

The activity reported in the next section will provide empirical evidence in support of these claims, while at the same time illustrating the variety of interesting explorations and discussions to which an analysis of errors can lead teachers.

III - Analyzing Incorrect Definitions of 'Circle':

An illustration

Overview of the Unit

Both the notions of 'circle' and of 'mathematical definition' are generally considered as non-problematic by mathematics teachers. I thought that an in-depth analysis of seemingly incorrect definitions of 'circle' could provide a concrete starting point for a discussion on the nature of mathematical definitions and of mathematics more generally, to be conducted with a group of mathematics teachers enrolled in one of my graduate education courses [1].

In order to introduce potentially erroneous definitions of 'circle', I searched for a group of students whose mathematical perceptions would be at odds with those of the secondary school teachers attending the course. In this case, a group of college students who were taking a remedial mathematics course was asked to recollect and/or to recreate a definition for the concept of 'circle' [2]. Below is the list produced by the college students and presented in its entirety to the group of mathematics teachers:

1. "Circle is a form in which radius is equal from the center to arc." (accompanied by a figure)
2. "A circle - a collection of points all equidistant from the center (radius)."
3. "Circle - has a center with a line around it, all the points on the line are an equal distance from the center, a circle is round."
4. "Circle - a closed round shape having the same radius throughout from the center."

5. "Definition of a circle: a perfectly round, closed figure with radius r and circumference c where r is the distance from the midpoint of the circle to any outside point and c is the distance measured around the outside."

6. "A circle is a set of point with a radius. Round thing."

7. "Circle - 1. a geometric form, 2.one dimensional, 3. a bent line with one end connected to another 4. a shape with no flat sides."

8. "A closed, continuous, rounded line."

9. "Circle: round, both ends meeting."

10. "Circle: a round object which has no beginning or end, which is smooth, and which has an infinitly number of points on it!!"

11. "Circle - a geometric figure which lies on a plain that consist of a line which begins and end at the same point."

12. "Def: of a Circle. A continuous round line, Infinite - all points from center of circle are equivalent to each other." (accompanied by a figure)

13. "Circle - 1. round 2. continuous - no beginning or end 3. a set of points such that when connected one gets a concoction called a circle."

14. "Circle: consecutive points in a 360 angle when connected is round and closed."

15. "Circle - closed line w/an angle of 360."

16. "Circle: $(x-h)^2 + (y-k)^2 = r$. Round"

17. "Circle = a set of coordinates falling on the plain
$(x-h)^2 + (y-k)^2 = r^2$ " (accompanied by a figure)

18. "Definition Circle: Circle is a straight line that changes directions constantly."

19. "Define 'circle' - something that is round - a round line like a orange, wheel."

20. "Round - 3.14 - shape of a orange, coin, earth - Pi."

21. "Circle - something whose area is = to πR^2.

22. "Circle : Includes all the points of the circumference and all the points inside it (plane)." (accompanied by a shaded figure)

One of the advantages of this list is that not only are there apparently clear-cut cases of correct and erroneous definitions, but there are borderline cases as well. The latter proved to be particularly valuable as a resource for generating doubts not only about the validity of even already accepted definitions of circle, but of the function and purpose of definitions more generally.

The analysis of these definitions of 'circle' involved three major activities, which evolved over four class periods and included two written homework assignments:

a) A preliminary analysis of the list of definitions produced by the remedial students, motivated by the task of classifying them. This involved a discussion in small groups, followed by a report to the rest of the class and a general discussion.

b) A mathematical study of the concept of 'circle', motivated by a categorization based on the mathematical properties of circles considered in each definition in the list and led by the instructor.

c) An inquiry into the nature of mathematical definitions and the "errors" that can occur in such a context, led by the evaluation of specific definitions in the list.

Each of these three aspects of the unit contributed to stimulate reflections on the nature of mathematics, in a more or less explicit and direct way.

For example, one group decided to categorize the definitions according to whether "they were acceptable or not" (without even

questioning whether such a decision would be problematic). The heated debate which generated over definition #21, however, revealed that mathematics teachers themselves may not always agree on whether a specific definition is correct or not. In turn, this unexpected result challenged the view implicitly held by some of the participants that mathematical results should always be either right or wrong, and motivated a more explicit identification of the criteria mathematicians employ to evaluate definitions.

The analysis of the properties of circle taken into consideration by each definition (whether correct or not) was also an eye-opening experience. Not only did this lead the class to discover the existence of quite different yet equally correct definitions of 'circle', it also showed that the evaluation of a specific definition would depend essentially upon the mathematical context in which one operates. For example, in topology, definitions #8 and #11 well characterize circles!

The third part of the unit, however, is the one where errors were most explicitly employed to initiate a reflection on the nature of mathematics. I have thus chosen to report in more detail the results of this last component of the unit, to provide evidence of the richness and depth of the discussions generated by a study of errors. In doing so, it will be helpful to report some excerpts from specific assignments as well as from the diaries kept by each participant in the course, where comments on class activities as well as spontaneous thoughts were often recorded [3].

Inquirying about the Notion of Mathematical Definition

The unexpected richness and interest of the first two activities brought many of the participants to challenge their initial negative judgment of most of the definitions produced by the remedial students.

> I am having an hard time with my definition of definition. Intellectually, I agree with the majority of my group ... that a definition must be exact to be correct. ... However, the part of me that sees the group as well as the black and whites, is much more lenient. That part believes in what I call the "personal definition." That part of me is willing to take into account a student ability when judging his definitions of mathematical terms. If he is able to show a solid understanding of a term without a classic textbook definition, I find difficult to find him in error. I feel he has a firmer grasp of the subject matter than someone who is able to "parrot" a definition, but possesses no understanding of the concept. (Jill, diary, week #2)

> I would think that students find it difficult to verbalize a definition of a picture, and will in fact describe it by its properties or what they heard in class.
> But are verbatim theorems, postulates and definitions the best way to teach students about geometric shapes? Wasn't it interesting to observe the way the students interpreted the definitions? Shouldn't such observations make us, as teachers, be aware of our own techniques in presenting material? Perhaps we are the ones who aren't making ourselves clear enough to our students? (Sonia, diary, week #2)

The atmosphere of doubt thus generated was ideal to proceed to a more explicit discussion on the nature of mathematical definitions.

The students were asked, in a subsequent written assignment, to write down their opinions regarding the main attributes of mathematical definitions, and what errors could occur with regard to them. As it often happens in these cases, many people raised interesting and insightful issues about the nature of mathematical definitions, although they generally tended to look only at some aspects of the whole picture. The most interesting contributions were collected in a hand-out, which

was distributed in class as a basis for a discussion on the nature of mathematical definitions. In this class discussion we ended up with the following categories and observations, generated by the students themselves:

a) **Use of terminology:**
- Vague terms should be avoided.
- Only previously defined terms should be used.

b) **Isolation of the concept:**
- The definition should not apply to other objects than the ones we want to define.
- All instances of the concept should be included.
- All exceptions should be included.

c) **Essentiality:**
- No unnecessary information should be included.
- Avoid unnecessary properties.
- Do not mention or define unnecessary terms.
- Avoid the use of long, big words.

d) **Understandability:**
- The concepts and terms used should be appropriate to the level of the audience.
- It should not be too complicate "to decode."
- It should provide an image for the concept.

e) **Usefulness or Usability:**
- It can be easily used or applied.
- It can be easily remembered.
- It is useful in other points of mathematics.
- It is useful in applications.

In other words, we found ourselves sharing the view of mathematical definition "passed" to us by the community of mathematicians, even if the inclusion of categories such as 'understandability' and 'usefulness' allowed some educational concerns to surface.

Once confronted with the task of evaluating specific definitions on the basis of the criteria just stated, however, the class often split

and we had to realize that the notion of what constitutes "a good mathematical definition" was not as clear as it had first appeared.

For instance, how should the fact that many people forgot to mention the plane be interpreted? We doubted that the authors of definitions 1 and 2 did not know the difference between circles and spheres. Rather, we assumed that they did not feel the need to specify that the points should all belong to the same plane, because it was obvious to them, especially if they thought of defining 'circle' in the context of plane Euclidean geometry. Should we then consider such omission an 'error'?

An additional problematic element came into the picture when we considered the use and roles of definitions not only in mathematics, but in the context of mathematics instruction as well. For example, in evaluating a definition produced by a student, we felt the need of taking into account whether s/he was just "parroting" a memorized definition, or whether s/he understood the concept behind it. In the latter case, an imprecise but original definition might even be considered as a positive contribution - something we as teachers tend not to appreciate when we strive for premature mathematical precision.

How should we evaluate alternative definitions for our own classroom teaching? The fact that the students must be able to understand them becomes a priority here. For instance, however essential and elegant, none of us would have chosen the differential geometry definition for a high school geometry course!

At this point somebody even argued that the concern for essentiality and elegance, when presenting the definition of a new

concept, does not seem to make much sense. On the contrary, as a teacher one would want to provide the students with as much information and examples as possible about the concept in question. This remark made us reflect once more on the role of definitions in the teaching of mathematics. We were then able to identify the existence of some confusion between 'definition and a few other related but distinct notions. Such confusion had been at the basis of several of our problems in the previous evaluation of specific definitions. We reached partial clarification when we realized the difference between the 'name', 'symbol', 'definition', 'description', 'model' and 'prototypical example' of a given mathematical concept.

For a few people in class, the reflection on the nature of mathematical definitions did not end with this class discussion. Further observations appeared in the diary entries for the following weeks. It is particularly interesting to follow the development of one student's (Elisa) conception of definition throughout our activities in this unit.

Elisa started with a very clear idea of what a "mathematical definition" should be, at least from the mathematical point of view. The following quote comes from her response to our initial written assignment, where she was asked to state the characteristics of good mathematical definitions:

> A mathematical definition is a set of statements which completely describe the essential characteristics of a mathematical object, word, or concept. A good definition should use only previously defined terms or the generally accepted undefined terms with which one begins the investigation of any deductive system. It should encompass

> all possible objects in the class of objects defined. It should not contain any unnecessary information. It should not be redundant or circular. It certainly should not contain contradictions or vague descriptive phrases which are subject to different interpretations. A good definition is reversible. It can be expressed in prose or it can take the form of acceptable mathematical notation. It should use the most generally acceptable notation available.

The analysis of the definitions of 'circle' conducted in class, however, soon challenged her own conception of mathematical definition. The first evidence of this can be seen in the frustration and the doubts she experienced and reported in her journal, following our collective analysis of our list of definitions of 'circle':

> I found the exercise of trying to categorize a series of "mostly incorrect" definitions of circle very frustrating. It is difficult for me to get beyond the fact that they are incorrect. (journal entry, week 2)

> I continue to be frustrated by the exercise of going into depth about categories of definitions of circle. ...
> I found your comments and categorization of definitions of circle as well as those given by each group in class very interesting. The effect of class discussion and your handout was to expand my narrow conception of how to define a circle. It also raised many doubts in my mind concerning the whole question of adequate definitions. The task of making good definitions is complex. ...
> Your categories and descriptions reminded me that the appropriateness of a definition is dependent on the mathematical sophistication of the person for whom the definition is given. (journal entry, week 3)

The reflections on the nature of mathematical definitions stimulated by our class discussions and activities, finally led Elisa to a finer and more relativistic conception of the notion:

> I am still thinking about problems surrounding definitions.
> It seems to me that whenever we make mathematical definitions we are doing so, or should be doing so, in the context of certain undefined terms and previously defined terms which the 'definer' and the 'receiver of the definition' must previously agree. ...

> Even when the context is clear, it is difficult to give fully precise definitions which could never be misinterpreted because our language and our symbols are not precise enough, and because words and symbols are often used to represent a variety of things. In order for a definition to be consider accurate and complete, the 'definer' and the 'receiver of the definition' must be on the same wavelength in terms of the word and symbols being used in the definition. If these conditions hold, then we can start giving more qualifications to the definition, i.e. whether it is reversible, whether it contains too much or just enough information, etc. (journal entry, week 4)
>
> I am still thinking about definitions. It seems that in our class discussions about definition we often use the word 'definition' when we really mean 'description'. It is not necessarily true that a good mathematical definition must give a good image or understanding of a concept. It is most often the case that the understanding of the concept comes before the technical definition can be understood. Teachers need to get concepts across by using many different descriptions and examples which illustrate those concepts before they give a precise definition. As was pointed out in class, long before precise definitions of geometric shapes are given, children are taught to recognize these shapes and have a good, intuitive understanding of their properties. It is a common practice in teaching mathematics to try to give students some feeling of a concept before handing out the definition. For us as teachers, it is often much more important that our students understand and can use definitions correctly than that they can verbalize them precisely. Possibly, we would have more success at stating the attributes of a good definition if we would differentiate carefully between what we mean by description and what we mean by definition. (journal entry, week 5)

Participants' Comments on the Activity

Though not everybody in class went to the same depth in their analysis of mathematical definitions, the experience at least started some doubt concerning their original conceptions of this notion. As revealed in the participants' responses to a questionnaire at the end of the coursse, most people found the activity rewarding, though demanding and time consuming. The following comments have been selected to

illustrate the teachers' explicit comments and evaluation of the experience:

> The definitions of circle were an eye-opening experience for me. It clearly helped me to see my feelings and understanding about the nature of definitions in mathematics.
>
> The definitions of circles was a very good activity, it gave all of us a chance to see just how it is each of us percieves such "obvious" concepts.
>
> When we began this exercise, I merely saw most of the definitions presented as being wrong, period! As we continued to discuss definitions, my understanding of the difficulties they present was expanded. I now have a heathier respect for the difficulty of presenting adequate definitions. I still feel, though, that people have a tendency to confuse the concepts of "description" and "definition".
>
> I became a little bored with the discussion of definitions of circle. However, it did generate a lot of thought. I still am trying to decide what a good definition consists of. If nothing else, it emphasized that an error to one is not necessarily an error to another.

IV - Other Applications of the Strategy

The experience described in the previous section illustrates how errors have the potential to engage teachers in valuable reflections and discussions on the nature of mathematics. At the same time, it may be important to provide further evidence that this success was not isolated, and that the value of using errors as springboards is not limited to this specific application regarding the notion of definition. In what follows, I will thus briefly present a few other examples of activities based on the in-depth study of errors, which were implemented in various teacher education courses. Details about the contexts in which each experience occurred, as well as references where the example is more fully developed, will be provided in notes.

First of all, I would like to mention that the same unit (with minor variations to respond to diverse audiences and time constraints) was implemented successfully in a few other contexts--within the same course offered at a different private university, and in methods' courses offered to pre-service mathematics teachers both at the same State university and private university [4]. In each case, the class discussions as well as individual written assignments confirmed the potential of an analysis of incorrect definitions of 'circle' to make people question the criteria used to evaluate mathematical definitions, become aware of the importance of the context in this evaluation, and realize that some of their expectations about the nature of mathematical definitions were rather unrealistic.

Other errors were also employed successfully in teacher education courses to examine general mathematical notions other than 'definition', and challenge the participants' expectations with respect to the "truth" and "rigor" of mathematical results.

For example, on two occasions [5] teachers were engaged in an in-depth analysis of the error contained in the following proof, suggested by the instructor to justify the result $\sqrt{2+\sqrt{2+\ldots+\sqrt{2+\sqrt{2}}}}=2$:

Let's call 'x' the expression we want to evaluate:

$$\sqrt{2+\sqrt{2+\ldots+\sqrt{2+\sqrt{2}}}}=x$$

squaring both sides, we get:

$$2+\sqrt{2+\ldots+\sqrt{2+\sqrt{2}}}=x^2$$

The radical on the left side is again x, since it also contains infinitely many terms. Therefore, we get the second degree equation:

$$2+x=x^2$$

Solving for x we get the solutions x=-1 and x=2. The root x=-1 has to be excluded, since the radical is certainly positive. Therefore it must be:

$$\sqrt{2+\sqrt{2+\ldots+\sqrt{2+\sqrt{2}}}}=2.$$

The realization that this "proof" was not rigorous enough because of some "missing steps" and yet produced both the correct result and a procedure which once refined would allow the construction of an acceptable proof [6], stimulated an interesting discussion on mathematical proofs. Many of the teachers involved declared their confusion, as college students, about what was necessary to write down in order to make a proof acceptable--since one could not put down explicitly every single step in the deduction, and yet it was difficult to decide what could be assumed as "obvious." They also admitted that

in their experience as mathematics students, they were rarely asked to produce original proofs, but rather were merely required to understand (and occasionally memorize) given proofs. The discussion revealed, once again, different positions and viewpoints amongst the participants on important mathematical issues. In addition to a clarification of established logical requirements for mathematical proofs, this activity allowed the participants to become more sensitive to the role of other elements (such as context and culture) in deciding whether a specific proof is indeed sufficiently "convincing".

The existence of limitations inherent to mathematics itself was made evident by a discussion of common errors encountered in the enumeration of infinite sets--such as the expectation that the natural numbers and their squares have a different cardinal number, since the latter is a proper subset of the former, or that the points in a segment should not be "more" than the natural numbers, since the latter set is unbounded [7]. A look at the debates and confusion generated around these issues in the history of mathematics revealed that not only had great mathematicians been troubled by the same unintuitive results as ourselves, but more importantly, that the current theory of cardinal and ordinal numbers has just one of the possible choices to resolve the controversy and that equally acceptable (though perhaps not equally useful and interesting) alternatives were possible. This realization seemed to have a considerable impact on the original view of mathematics held by many of the participants [8].

To a minor extent, the discussion of the imaginary number 'i' is neither greater than, nor less than, nor equal to zero, and its

implications, also provided valuable insights on the limitations mathematicians sometimes have to accept when trying to extend a familiar system--contrary to the implicit expectation that "bigger means better"--and helped the participants better understand the principles and limitations of proofs by contradiction [9].

The comparison of the following "errors" has proved useful in starting a discussion about the role of "exceptions" in mathematics [9]:

$$1)\quad \frac{10a+b}{10b+c} = \frac{a}{c}\ ,\ \text{since}\ \frac{16}{64} = \frac{1}{4}.$$

$$2)\quad \frac{1}{a}-\frac{1}{b} = \frac{1}{ab}\ ,\ \text{since}\ \frac{1}{2}-\frac{1}{3} = \frac{1}{6}\ ,\ \frac{1}{3}-\frac{1}{4} = \frac{1}{12}\ ,\ \text{etc.}$$

$$3)\quad \frac{a \times b}{b \times c} = \frac{a}{c}\ .$$

In each of these cases, the general rule is only partially correct, yet the number of exceptions to each makes the three situations quite different. This brought us to discuss also the role of partial results and conjectures, and to inquire into what happened historically when exceptions to established conjectures were discovered at a later time--something that did occur on many occasions, and often provided a catalyst for important realizations and occasionally even opened new areas for research (as for example in number theory).

Conclusions

The experiences reported in this paper suggest that the in-depth study of mathematical errors can be of great value for teacher education, by eliciting teachers' beliefs about the nature of the discipline, by revealing the existence of alternative viewpoints and thus fostering genuine dialogue, and by providing a concrete starting point for reflections on abstract and complex issues.

Though the examples discussed are merely a small selection of the variety of creative activities and discussions that can be stimulated by an analysis of errors, I hope they have provided sufficient evidence of the potential of using errors as "springboards" to encourage others to apply this strategy with different errors and in various contexts.

Footnotes

[1] The experience reported in this paper was conducted in a course, entitled "Educational Uses of Errors", taught by the author at the State University of New York at Buffalo (USA) in Fall 1984. 15 graduate students in mathematics education programs enrolled in the course, they were mostly mathematics teachers, at either secondary school or college level.

[2] These definitions were collected by Mr. B.Reopolos, a graduate student assisting in the course. He asked his students', enrolled in a pre-calculus course offered by the Learning Center of the State University of New York at Buffalo (USA) in Spring 1984.

[3] An important component of the course was the creation of a "class journal". Each of the participants in the course was expected to contribute a weekly entry, which would be reproduced and distributed to the rest of the class. The topic of the entry could be anything related to the course theme, and often students included their reactions to activities conducted in class, offered further contributions to on-going discussions, reported personal experiences and thoughts related to errors and mathematics.

[4] These experience occurred in the following courses, respectively: the course "Educational Uses of Errors", offered at the University of Rochester, Rochester, NY, in Spring 1986 to 12 graduate students in mathematics education programs; the course "Methods of Teaching Secondary Mathematics", offered at the State University of New York at Buffalo, Buffalo, NY, in Spring 1985 to 14 pre-service mathematics teachers; the course "Seminar I in the Learning and Teaching of Mathematics", offered at the University of Rochester, Rochester, NY, in Fall 1986 to 13 pre-service mathematics teachers.

[5] This unit was used in both implementations of the above mentioned course "Educational Uses of Errors", at SUNY/Buffalo and at the University of Rochester.

[6] A thorough study of this incorrect proof and the results of explorations stimulated by it, can be found in Borasi [1985a].

[7] This experience occurred as part of the course "Educational Uses of Errors" offered at SUNY/Buffalo.

[8] An analysis of the difficulties, errors and debates generated around the problem of enumerating infinite sets can be found in Borasi [1985b].

[9] The limitations inherent to the set of complex numbers has been discussed with varying depth in all of the four courses mentioned in this paper.

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